General Certificate of Education
Advanced Subsidiary (AS) and Advanced Level

MATHEMATICS **M4**

Mechanics 4

Additional materials:
Answer paper
Graph paper
List of Formulae

SPECIMEN PAPER

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.

Where a numerical value for the acceleration due to gravity is needed, use 9.8 m s^{-2}.

You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 60.

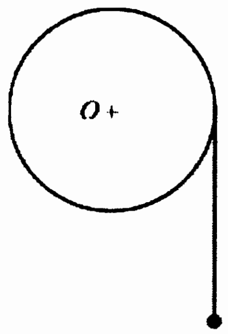
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

You are reminded of the need for clear presentation in your answers.

This question paper consists of 4 printed pages.

1 The region bounded by the part of the curve $y = \sqrt{x}$ from $x = 0$ to $x = a$, the x-axis and the line $x = a$ is rotated completely about the x-axis to form a uniform solid of revolution. Find by integration the x-coordinate of the centre of mass of the solid. **[6]**

2



The diagram shows a uniform circular disc, of mass m and radius a, which is free to rotate in a vertical plane about a smooth fixed horizontal axis through its centre O. A light inextensible string is wrapped round the circumference of the disc, and has one end attached to the circumference. A particle of mass m hangs freely at the other end of the string. The system is released from rest. Show that the angular acceleration of the disc is $\frac{2g}{3a}$, and find the tension in the string. **[6]**

3 A rigid square frame consists of four uniform rods, each of mass m and length $2a$, joined at their ends to form a square. Show that the moment of inertia of the frame, about an axis through one of its corners and perpendicular to its plane, is $\frac{40}{3}ma^2$. **[3]**

The frame is suspended from one corner, and can rotate in a vertical plane about a smooth horizontal axis through that corner. Show that the motion in which the frame makes small oscillations about its equilibrium position is approximately simple harmonic, and find the period of this simple harmonic motion. **[5]**

4 A uniform circular disc, of mass m and radius a, can rotate in a vertical plane about a fixed horizontal axis passing through its centre O. When the disc is at rest, a particle of mass $2m$ is released, from rest, at a height $2a$ vertically above one end A of the horizontal diameter of the disc. The particle falls freely, strikes the disc at A, and adheres to the disc. Find the angular speed with which the disc starts to rotate. **[5]**

While the disc (with the attached particle) rotates, a constant frictional couple C acts on the disc. The disc comes to rest after one quarter of a revolution, when the particle is at the lowest point of the disc. Find C. **[3]**

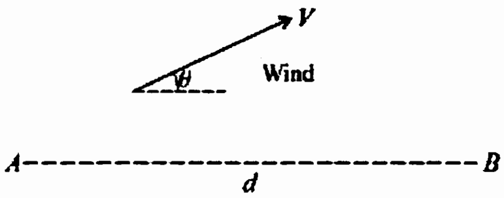
5 A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a smooth horizontal axis through A. When the rod is hanging at rest in equilibrium with B vertically below A, it is given an angular speed Ω, where

$$\Omega^2 = \frac{3g}{4a}.$$

Show that the rod comes instantaneously to rest when it has turned through an angle $\frac{1}{3}\pi$. [2]

At this position of instantaneous rest, the force acting on the rod at A has horizontal and vertical components X and Y respectively. Find X and Y in terms of m and g. [8]

6



A light aircraft flies flies from A due east to B and then flies directly back to A. The distance AB is d, and the speed of the aircraft relative to the air is $4V$. During the flight, there is a steady wind, of speed V in a direction making an angle θ with AB (see diagram). Show that the total flying time for the journey from A to B and back is

$$\frac{2d\sqrt{15+\cos^2\theta}}{15V}.$$ [10]

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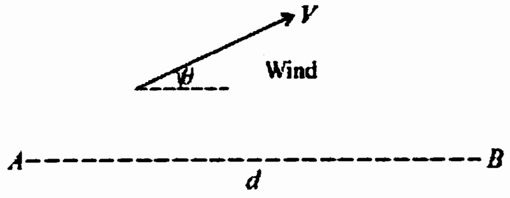

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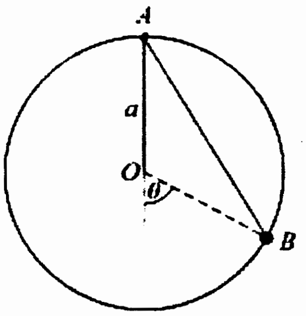


A small smooth bead B, of mass m, is threaded on a circular wire with centre O and radius a. The wire is fixed in a vertical plane. A light elastic string, of natural length a and modulus of elasticity λ, has one end fixed at O. The string passes through a small smooth ring A fixed at the highest point of the wire, and the other end of the string is attached to B. The diagram shows the system at an instant when OB makes an angle θ with the downward vertical at O.

(i) Taking the horizontal level of O as the reference level for gravitational potential energy, show that the total potential energy of the system in the position shown is

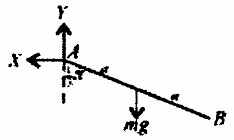
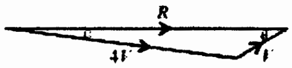
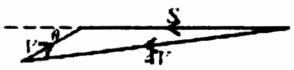
$$\lambda a + a(\lambda - mg)\cos\theta \ .$$ **[4]**

(ii) Hence show that there is a position of stable equilibrium with $\theta = 0$ so long as $\lambda < mg$. **[3]**

(iii) Given that $\lambda = \frac{1}{2}mg$, show that the approximate period of small oscillations about the equilibrium position is

$$2\pi\sqrt{\frac{2a}{g}}\ .$$ **[5]**

1 Total mass $= \pi\rho\int_0^a x\,dx$	M1	For relevant use of $\int y^2\,dx$
$= \frac{1}{2}\pi\rho a^2$	A1	Introduction of ρ is required
Total moment $= \pi\rho\int_0^a x^2\,dx$	M1	For relevant use of $\int xy^2\,dx$
$= \frac{1}{3}\pi\rho a^3$	A1	No further penalty if ρ omitted
Hence $\frac{1}{2}\pi\rho a^2\bar{x} = \frac{1}{3}\pi\rho a^3 \Rightarrow \bar{x} = \frac{2}{3}a$	M1	For equating and solving for $\bar{x}$
	A1 6	For correct final answer
2 For particle: $mg - T = mf$	B1	
For disc: $Ta = \frac{1}{2}ma^2\alpha$	B1	
String moves with disc, so $f = a\alpha$	B1	Stated or used at any stage
Hence $mg = mf + \frac{1}{2}ma\alpha = \frac{3}{2}ma\alpha$	M1	For obtaining an equation in α (or T)
$\alpha = \dfrac{2g}{3a}$	A1	Given answer shown correctly
$T = \frac{1}{3}mg$	A1 6	
3 M.I. for one of the rods opposite the axis is:	M1	Relevant use of parallel axes theorem
$\frac{1}{3}ma^2 + m(a^2 + 4a^2) = \frac{16}{3}ma^2$	A1	For correct unsimplified expression
M.I. of frame $= 2(\frac{4}{3}ma^2 + \frac{16}{3}ma^2) = \frac{40}{3}ma^2$	A1 3	Given answer shown correctly
Weight $4mg$ acts at $a\sqrt{2}$ from the axis	B1	
Equation of motion is: $4mga\sqrt{2}\sin\theta = -\frac{40}{3}ma^2\ddot{\theta}$	M1	For relevant use of $C = I\ddot{\theta}$ at *general* posn
	A1	Correct equation
i.e. $\ddot{\theta} \approx -\dfrac{3g\sqrt{2}}{10a}\theta$	A1	Reduction to standard SHM form
Hence SHM with period $2\pi\sqrt{\dfrac{10a}{3g\sqrt{2}}}$	A1✓ 5	
4 Speed of particle before impact is $\sqrt{4ga}$	B1	
$2m\sqrt{4ga} \times a = (\frac{1}{2}ma^2 + 2ma^2)\omega$	M1	Equating ang mom before and after
	B1	For total M.I. $(\frac{1}{2}ma^2 + 2ma^2)$, or equivalent
	A1	Correct equation
Hence $\omega = \dfrac{8}{5}\sqrt{\dfrac{g}{a}}$	A1 5	
Work done against friction is $C \times \frac{1}{2}\pi$	B1	
Hence $\frac{1}{2}\pi C = \frac{5}{4}ma^2\omega^2 + 2mga$	M1	Equating energy (KE and PE) to work
$C = \dfrac{52mga}{5\pi}$	A1✓ 3	

5	$\frac{1}{2}\cdot\frac{4}{3}ma^2.\frac{3g}{4a} = mga(1-\cos\theta)$	M1		For relevant use of conservation of energy
	LHS $= \frac{1}{2}mga$, so $\theta = \frac{1}{3}\pi$	A1	2	Given answer obtained or verified correctly
	 			They may have X and/or Y reversed; any consistent work is acceptable
	$mga\sin(\frac{1}{3}\pi) = \frac{4}{3}ma^2\alpha$	M1		Taking moments about A to find ang acc
	i.e. $\alpha = \frac{3\sqrt{3}g}{8a}$	A1		
	Acceleration component of CG || rod is zero	B1		May be implied
	and perpendicular to rod is $\frac{3\sqrt{3}g}{8}$	A1✓		
	EITHER : Res hor: $X = m.\frac{3\sqrt{3}g}{8}\cos(\frac{1}{3}\pi)$	M1		To include attempt at resolving transverse acc
	i.e. $X = \frac{3\sqrt{3}}{16}mg$	A1		
	Res vert: $mg - Y = m.\frac{3\sqrt{3}g}{8}\sin(\frac{1}{3}\pi)$	M1		
	Hence $Y = \frac{7}{16}mg$	A1		
	OR : Res || rod: $\frac{1}{2}\sqrt{3}X + \frac{1}{2}Y = \frac{1}{2}mg$	B1		
	Res ⊥ rod:	M1		4 terms required
	$\frac{1}{2}X - \frac{1}{2}\sqrt{3}Y + \frac{1}{2}\sqrt{3}mg = \frac{3}{8}\sqrt{3}mg$	A1		Correct equation
	$X = \frac{3\sqrt{3}}{16}mg,\ Y = \frac{7}{16}mg$	A1	8	For both answers
6	 	B1		For correct triangle; may be implied
	$\sin\alpha = \frac{1}{4}\sin\theta$ or $16V^2 = R^2 + V^2 - 2RV\cos\theta$	B1		For appropriate use of sine or cosine rule
	$R = V\cos\theta + 4V\sqrt{1-\frac{1}{4}\sin^2\theta}$ or	M1		Any method for R in terms of V and θ
	$R = V\cos\theta + V\sqrt{\cos^2\theta + 15}$ or equivalent	A1		For any correct (unsimplified) expression
	 	B1		For correct return triangle; may be implied
	$S = -V\cos\theta + V\sqrt{\cos^2\theta + 15}$ or equivalent	M1		Any method for S in terms of V and θ
		A1		For any correct (unsimplified) expression
	Total time is $\frac{d}{R} + \frac{d}{S}$	M1		For expressing this in terms of d, V, θ
	i.e. $\frac{d(R+S)}{RS} = \frac{2dV\sqrt{\cos^2\theta + 15}}{V^2(\cos^2\theta + 15) - V^2\cos^2\theta}$	M1		For combining the terms algebraically
	$= \frac{2d\sqrt{15 + \cos^2\theta}}{15V}$	A1	10	Given answer correctly shown

7 (i) $AB = 2a\cos\frac{1}{2}\theta$ or $AB^2 = 2a^2 + 2a^2\cos\theta$	M1		
Elastic energy $= \dfrac{2a^2\lambda(1+\cos\theta)}{2a}$	A1		Or equivalent
PE of B is $-mga\cos\theta$	B1		
Total energy $V = \lambda a + a(\lambda - mg)\cos\theta$	A1	4	Given answer correctly shown
(ii) $\dfrac{dV}{d\theta} = -a(\lambda - mg)\sin\theta = 0$	M1		Differentiate and equate to zero, or equivalent argument from properties of cos graph
Hence $\theta = 0$	A1		Given answer correctly shown
$\left.\dfrac{d^2V}{d\theta^2}\right\|_{\theta=0} = a(mg-\lambda) > 0$ if $\lambda < mg$	A1	3	Stability explained via identification of min V
(iii) $\frac{1}{2}m(a\dot{\theta})^2 + V =$ constant	B1		
$ma^2\ddot{\theta} + \frac{1}{2}mga\sin\theta = 0$	M1		Differentiate with respect to t
	A1		Correct simplified equation
i.e. $\ddot{\theta} \approx -\dfrac{g}{2a}\theta$, so SHM	M1		Reduction to standard form
Period is $2\pi\sqrt{\dfrac{2a}{g}}$	A1	5	Given answer correctly shown

